On ZK equation and generalizations

Problems in the dynamics of nonlinear dispersive equations

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Felipe Linares

IMPA

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In this talk we will consider the initial value problem associated to the nonlinear equation

$$\begin{cases} u_t + u\,\partial_x u + \partial_x \Delta u = 0, \\ u(x,y,0) = u_0(x,y) \end{cases} \tag{1}$$

called the Zakharov-Kuznetsov equation, where u is a real valued function defined in some suitable domain and $\Delta = \partial_x^2 + \partial_y^2$.

Outline

- Model
- Well-posedness for the ZK equation in a cylinder and on the background of a KdV soliton (A. Pastor and J-C. Saut)
  - Motivation and Previous Results
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- Generalized ZK equation (A. Pastor and L.G. Farah)
  - Local Results
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Model

The equation under consideration is a 2D version of the Zakharov-Kuznetsov equation, that is,

$$u_t + u\,\partial_x u + \partial_x \Delta u = 0, \tag{2}$$

This equation was first derived by Zakharov and Kuznetsov (1974) in three-dimensional form to describe nonlinear ion–acoustic waves in a magnetized plasma. A variety of physical phenomena, are governed by this type of equation; for example, the long waves on a thin liquid film, the Rosby waves in rotating atmosphere, and the isolated vortex of the drift waves in three-dimensional plasma.

Even though the Zakharov-Kuznetsov equation seems a natural generalization of the Korteweg-de Vries equation,

$$\partial_t v + v\partial_x v + \partial_x^3 v = 0,$$

the ZK equation is derived from the Euler-Poisson system for nonlinear ion-acoustic waves in a magnetized plasma.

$$\begin{cases} n_t + \operatorname{div}(nv) = 0 \\ v_t + (v \cdot \nabla)v + \nabla\varphi + a\, e_x \times v = 0 \qquad e_x = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T \\ \Delta\varphi - e^{\varphi} + n = 0 \end{cases}$$

where

$n =$ ion density $v =$ ion velocity $\varphi =$ electrostatic potential and

$a \geq 0$ measures the applied magnetic field.

ZK equation in a cylinder and on the background of a KdV soliton

Motivation

The Zakharov-Kuznetsov (ZK) equation admits as a solution the well-known KdV solitary wave $\phi_\omega(x,t) = \phi_\omega(x - \omega t)$, where

$$\phi_\omega(\xi) = 3\omega \text{sech}^2\left(\frac{\sqrt{\omega}}{2}\xi\right), \quad \omega > 0.$$

More generally, the N-soliton ϕ^N of the KdV equation is also a particular solution of the ZK equation which is smooth and bounded together with its time and space derivatives and behaves as a sum of solitons of velocities $4n^2, 1 \le n \le N$ when $t \to \infty$.

For instance, the 2-soliton ϕ^2 is given by

$$\phi^2(x,t) = 72\,\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}$$

A fundamental issue is that of the **transverse** stability/instability of those one-dimensional “localized” solutions of the KdV equation (such as the solitary wave) with respect to transverse perturbations governed by the ZK equation. This question was rigorously addressed recently by Rousset and Tzvetkov who developed a general theory which applies in particular to one-dimensional transverse perturbations of the KdV solitary wave.

Functional framework for the Cauchy problem which should be suitable to describe the aforementioned transverse perturbations.

This framework cannot be the classical Sobolev spaces $H^s(\mathbb{R}^d)$ since the KdV soliton or multi-soliton do not belong to this class of spaces. A natural space to study the transverse stability of localized one-dimensional solutions should contain those solutions.

A first possibility consists in functions which are “localized” in x and periodic in y. This leads to our study of the Cauchy problem for the ZK equation in $H^s(\mathbb{R} \times \mathbb{T})$.

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the one-dimensional torus. We will thus consider the IVP associated to the ZK equation in a cylinder

$$\begin{cases} \partial_t u + \partial_x \Delta u + u\partial_x u = 0, \quad (x,y) \in \mathbb{R} \times \mathbb{T}, \ \ t > 0, \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{3}$$

A second possibility is to consider two-dimensional "localized" perturbations of the one-dimensional solution ϕ. This motivates the study of the Cauchy problem,

$$\begin{cases} u_t + \partial_x \Delta u + u \partial_x u + \partial_x(\phi u) = 0, \quad (x, y) \in \mathbb{R}^2, \quad t > 0 \\ u(x, y, 0) = u_0(x, y), \end{cases} \tag{4}$$

where ϕ is the KdV solitary wave solitary wave or more generally any N-soliton of the KdV equation. Actually, we will only use that $\phi = \phi(x, t)$ is a solution of the KdV equation which is smooth and bounded together with its time and space derivatives, and furthemore belongs to the space $L^2_x L^\infty_T$. Those assumptions are obviously satisfied by the N-soliton solution of the KdV equation.

– Local well-posedness

We can see that if u is a solution of (1) with data u_0, then $u_\lambda(x,y,t) = \lambda u(\lambda x, \lambda y, \lambda^3 t)$ is also a solution with data $u_\lambda(x,y,0) = \lambda u_0(\lambda x, \lambda y)$. In particular, we have that

$$\|u_\lambda(0)\|_{\dot{H}^s(\mathbb{R}^2)} = \lambda^{s+1}\|u_0\|_{\dot{H}^s(\mathbb{R}^2)}.$$

This means that derivatives of the solutions remain invariant only if

$$s = -1.$$

This scaling argument suggests local well-posedness for $s \geq -1$.

– Global well-posedness

We note that the Zakharov-Kuznetsov equation has two conserved quantities, namely,

$$I_1(u(t)) = \int_{\mathbb{R}^2} u^2(x,y,t)\ dxdy = \int_{\mathbb{R}^2} u_0^2(x,y)\ dxdy,$$

and

$$\begin{aligned} I_2(u(t)) &= \frac{1}{2}\int_{\mathbb{R}^2} (u_x^2 + u_y^2 - \frac{1}{3}u^3)(x,y,t)\ dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^2} (u_{0x}^2 + u_{0y}^2 - \frac{1}{3}u_0^3)(x,y)\ dxdy. \end{aligned}$$

Previous Results

- Faminskii (1995) Local and Global well-posedness for ZK equation for data in $H^1(\mathbb{R}^2)$.
- Biagioni-L (2003) Local and Global well-posedness for modified ZK equation for data in $H^1(\mathbb{R}^2)$.
- L-Saut (2008) Local well-posedness for ZK equation in 3D for data in $H^{3/2+}(\mathbb{R}^3)$.
- L-Pastor (2009) Local well-posedness for ZK and mZK equation in 2D for data in $H^s(\mathbb{R}^2)$, $s > 3/4$.

The notion of well-posedness we use is the one given by Kato, that is, existence, uniqueness, persistence property and continuous dependence upon the data.

Main Results

Theorem 1 (ZK on a cylinder) *Given* $u_0 \in X^s, s > 3/2$, *there exist* $T = T(\|u_0\|_{X^s})$ *and a unique solution* u *of the IVP* (3), *such that* $u \in C([0,T] : X^s)$ *and* $u, \partial_x u, \partial_y u \in L^1_T L^\infty_{xy}$. *Moreover, the map data-solution* $u_0 \in X^s \mapsto u \in C([0,T] : X^s)$ *is continuous.*

The functional space X^s is defined by the norm

$$\|f\|_{X^s} := \|J_x^s f\|_{L^2(\mathbb{R}\times\mathbb{T})} + \|J_y^s f\|_{L^2(\mathbb{R}\times\mathbb{T})}. \tag{5}$$

Theorem 2 (Perturbed ZK) *For any* $u_0 \in H^1(\mathbb{R}^2)$*, there exist* $T = T(\|u_0\|_{H^1}) > 0$ *and a unique solution of the IVP* (4)*, defined in the interval* $\mathbb{R}^+$*, such that for any* $T > 0$

$$u \in C(\mathbb{R}^+; H^1(\mathbb{R}^2)), \tag{6}$$

$$\|\partial_x^2 u\|_{L_x^\infty L_{yT}^2} + \|\partial_y \partial_x u\|_{L_x^\infty L_{yT}^2} < \infty, \tag{7}$$

$$\|\partial_x u\|_{L_T^2 L_{xy}^\infty} < \infty, \tag{8}$$

and

$$\|u\|_{L_x^2 L_{yT}^\infty} < \infty. \tag{9}$$

Moreover, there exists a neighborhood W of u_0 in $H^1(\mathbb{R}^2)$ such that the map $\widetilde{u}_0 \mapsto \widetilde{u}(t)$ from W into the class defined by (6)–(9) is smooth. One has the two energy identities

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{R}^2} u^2(x,y,t)dxdy\right] + \frac{1}{2}\int_{\mathbb{R}^2} u^2(x,y,t)\phi(x,t)dxdy = 0 \tag{10}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left[\int_{\mathbb{R}^2}(|\nabla u|^2 - \frac{1}{3}u^3)(x,y,t) - u^2(x,y,t)\phi(x,t)dxdy\right]\\ &+\frac{1}{2}\int_{\mathbb{R}^2} u^2(x,y,t)\phi_t(x,t)dxdy = 0.\end{aligned} \tag{11}$$

Ingredients and ideas of the proofs

ZK on a cylinder

To deal with the IVP associated to (3) we will follow the ideas introduced by Ionescu and Kenig to study the IVP associated to the KP I equation in a cylinder and in $\mathbb{T}^2$. Their argument of proof is an extension of the methods used by Koch and Tzvetkov for the Benjamin-Ono equation, Kenig and König also for the Benjamin-Ono equation and by Kenig for the KPI equation.

– Energy estimates

Lemma 1 *Let u be a solution of the IVP* (3) *with $u_0 \in H^\infty(\mathbb{R} \times \mathbb{T})$. Then for any $s \geq 1$, we have, for any $T \in [0,1]$, that*

$$\sup_{0<t<T} \|u(t)\|_{X^s} \leq c_s \exp\left(c_s \left(\|u\|_{L^1_T L^\infty_{xy}} + \|\partial_x u\|_{L^1_T L^\infty_{xy}} + \|\partial_y u\|_{L^1_T L^\infty_{xy}}\right)\right) \|u_0\|_{X^s}, \tag{12}$$

where

$$\|f\|_{X^s} = \|J_x^s f\|_{L^2(\mathbb{R}\times\mathbb{T})} + \|J_y^s f\|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

Kato-Ponce commutators: Let $s \geq 1$ and $f, g \in H^\infty(\mathbb{R} \times \mathbb{T})$. Then

$$\|J_y^s(fg) - f J_y^s g\|_{L^2_y} \leq C_s \left[\|J_y^s f\|_{L^2_y} \|g\|_{L^\infty_y} + (\|f\|_{L^\infty_y} + \|\partial_y f\|_{L^\infty_y}) \|J_y^{s-1} g\|_{L^2_y}\right].$$

– A priori estimate

Lemma 2 *Let* u *be a solution of* (3) *with* $u_0 \in H^\infty(\mathbb{R} \times \mathbb{T})$ *defined in* $[0,T]$. *Let*

$$|||u_0|||_{X^s} = \|J_x^s u_0\|_{L^2(\mathbb{R}\times\mathbb{T})} + \|J_y^s u_0\|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

Then for any $s > 3/2$, *there exists* $T = T(\|u_0\|_{X^s}, s)$ *and a constant* $c_T(\|u_0\|_{X^s}, s)$ *such that*

$$f(T) := \int_0^T \Big(\|u(t)\|_{L^\infty_{xy}} + \|\partial_x u(t)\|_{L^\infty_{xy}} + \|\partial_y u(t)\|_{L^\infty_{xy}} \Big) dt \le c_T. \qquad (13)$$

– Key estimate

Lemma 3 *Let* $u \in C([0,T] : H^\infty(\mathbb{R} \times \mathbb{T})) \cap C^1([0,T] : H^\infty(\mathbb{R} \times \mathbb{T}))$*,* $F \in C([0,T] : H^\infty(\mathbb{R} \times \mathbb{T}))$*,* $T \in [0,1]$*, and*

$$\partial_t u + \partial_x \Delta u = \partial_x F \quad \textit{on} \quad \mathbb{R} \times \mathbb{T} \times [0,T]. \tag{14}$$

Then, for any $s_1 > 0$ *and* $s_2 > 1/2$*,*

$$\|u\|_{L^1_T L^\infty_{xy}} \leq c_{s_1, s_2} T^{1/2} \Big(\| J_x^{s_1} J_y^{s_2} u \|_{L^\infty_T L^2_{xy}} + \| J_x^{s_1} F \|_{L^1_T L^2_{xy}} \Big). \tag{15}$$

– Localized Strichartz estimate

Consider the solution of the linear IVP

$$\begin{cases} \partial_t u + \partial_x \Delta u = 0, & (x,y) \in \mathbb{R} \times \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{16}$$

that is, $u(x,y,t) = W(t)u_0(x,y)$, where

$$W(t)u_0(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(x\xi + yn + t(\xi^3 + \xi n^2))} \widehat{u}_0(\xi, n)\, d\xi. \tag{17}$$

Then

Lemma 4 *Assume* $\phi \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$. *Then, for any* $\epsilon > 0$,

$$\|W(t)Q_y^k Q_x^j \phi\|_{L^2_{2^{-k-j}} L^\infty_{xy}} \leq C_\epsilon 2^{(-1/2+\epsilon)j} \|Q_y^k Q_x^j \phi\|_{L^2_{xy}}. \tag{18}$$

The operators Q_x^k and Q_y^k on $H^\infty(\mathbb{R} \times \mathbb{T})$ are defined by

$\widehat{Q_x^0 g}(\xi, n) = \chi_{[0,1)}(|\xi|)\widehat{g}(\xi, n), \quad \widehat{Q_x^k g}(\xi, n) = \chi_{[2^{k-1}, 2^k)}(|\xi|)\widehat{g}(\xi, n)$ if $k \geq 1$,

and

$\widehat{Q_y^0 g}(\xi, n) = \chi_{[0,1)}(|n|)\widehat{g}(\xi, n), \quad \widehat{Q_y^k g}(\xi, n) = \chi_{[2^{k-1}, 2^k)}(|n|)\widehat{g}(\xi, n)$ if $k \geq 1$.

Perturbed ZK

Consider the linear IVP

$$\begin{cases} u_t + \partial_x \Delta u = 0, \qquad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \\ \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{19}$$

The solution of (19) is given by the unitary group $\{U(t)\}_{t=-\infty}^{\infty}$ such that

$$\begin{aligned} u(t) &= U(t)u_0(x,y) \\ &= \int_{\mathbb{R}^2} e^{i(t(\xi^3+\xi\eta^2)+x\xi+y\eta)}\,\widehat{u}_0(\xi,\eta)d\xi d\eta. \end{aligned}$$

Proposition 1 *Let* $0 \leq \varepsilon < 1/2$ *and* $0 \leq \theta \leq 1$*. Then the group* $\{U(t)\}_{t=-\infty}^{\infty}$ *satisfies*

$$\|D_x^{\theta\varepsilon/2} U(t) f\|_{L_t^q L_{xy}^p} \leq c \|f\|_{L_{xy}^2},$$

$$\|D_x^{\theta\varepsilon} \int_{-\infty}^{\infty} U(t-t') g(\cdot, t') dt' \|_{L_t^q L_{xy}^p} \leq c \|g\|_{L_t^{q'} L_{xy}^{p'}},$$

$$\|D_x^{\theta\varepsilon} \int_{-\infty}^{\infty} U(t) g(\cdot, t) dt \|_{L_{xy}^2} \leq c \|g\|_{L_t^{q'} L_{xy}^{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ *with*

$$p = \frac{2}{1-\theta} \quad \text{and} \quad \frac{2}{q} = \frac{\theta(2+\varepsilon)}{3}.$$

As a consequence of Proposition 1 we have

Let $0 \le \varepsilon < 1/2$. Then the group $\{U(t)\}_{t=-\infty}^{\infty}$ satisfies

$$\|U(t)f\|_{L_T^2 L_{xy}^\infty} \le cT^\gamma \|D_x^{-\varepsilon/2} f\|_{L_{xy}^2} \tag{20}$$

and

$$\|U(t)f\|_{L_T^{9/4} L_{xy}^\infty} \le cT^\delta \|D_x^{-\varepsilon/2} f\|_{L_{xy}^2}, \tag{21}$$

where $\gamma = (1-\varepsilon)/6$ and $\delta = (2-3\varepsilon)/18$.

Smoothing Effect

Lemma 5 *Let* $u_0 \in L^2(\mathbb{R}^2)$. *Then,*

$$\|\partial_x U(t) u_0\|_{L^\infty_x L^2_{yT}} \leq c\|u_0\|_{L^2_{xy}}.$$

Maximal Function Estimate

Lemma 6 *Let* $u_0 \in H^s(\mathbb{R}^2)$, $s > 3/4$. *Then,*

$$\|U(t)u_0\|_{L^2_x L^\infty_{yT}} \leq c(s,T)\|u_0\|_{H^s_{xy}},$$

where $c(s,T)$ *is a constant depending on* s *and* T.

Consider the integral operator

$$\Psi(u)(t) = \Psi_{u_0}(u)(t)$$
$$= U(t)u_0 + \int_0^t U(t-t')(u^2u_x + \partial_x(\phi u))(t')dt'$$

and define the metric spaces

$$\mathcal{Y}_T = \{u \in C([0,T]; H^s(\mathbb{R}^2)); \quad |||u||| < \infty\}$$

and

$$\mathcal{Y}_T^a = \{u \in \mathcal{X}_T; \quad |||u||| \leq a\},$$

with

$$|||u||| := \|u\|_{L_T^\infty H_{xy}^1} + \|\partial_x u\|_{L_T^2 L_{xy}^\infty} + \|\partial_x^2 u\|_{L_x^\infty L_{yT}^2} + \|\partial_{xy}^2 u\|_{L_x^\infty L_{yT}^2} + \|u\|_{L_x^2 L_{yT}^\infty},$$

where $a, T > 0$ to be determined.

Generalized Zakharov-Kuznetsov equation

Next we consider the IVP

$$\begin{cases} u_t + \partial_x \Delta u + u^k u_x = 0, & (x,y) \in \mathbb{R}^2, \quad t > 0, \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{22}$$

- For $2 \leq k \leq 7$, the IVP above is shown to be locally well-posed for data in $H^s(\mathbb{R}^2)$, $s > 3/4 \quad (3/4 > s_{\text{scal}} = 1 - \frac{2}{k})$.
- For $k \geq 8$, local well-posedness is shown to hold for data in $H^s(\mathbb{R}^2)$, $s > s_k$, where $s_k = 1 - \frac{3}{2(k-2)} \quad (s_k \geq s_{\text{scal}} = 1 - \frac{2}{k})$.

Theorem 3 *Assume* $3 \leq k \leq 7$. *For any* $u_0 \in H^s(\mathbb{R}^2)$, $s > 3/4$, *there exist* $T = T(\|u_0\|_{H^s}) > 0$ *and a unique solution of the IVP* (22), *defined in the interval* $[0, T]$, *such that*

$$u \in C([0,T]; H^s(\mathbb{R}^2)),$$
$$\|D_x^s u_x\|_{L_x^\infty L_{yT}^2} + \|D_y^s u_x\|_{L_x^\infty L_{yT}^2} < \infty, \tag{23}$$
$$\|u\|_{L_T^{p_k} L_{xy}^\infty} + \|u_x\|_{L_T^{12/5} L_{xy}^\infty} < \infty,$$

and

$$\|u\|_{L_x^4 L_{yT}^\infty} < \infty, \tag{24}$$

where $p_k = \frac{12(k-1)}{7-12\gamma}$ *and* $\gamma \in (0, 1/12)$. *Moreover, for any* $T' \in (0, T)$ *there exists a neighborhood* W *of* u_0 *in* $H^s(\mathbb{R}^2)$ *such that the map* $\widetilde{u}_0 \mapsto \widetilde{u}(t)$ *from* W *into the class defined by* (23)–(24) *is smooth.*

Theorem 4 *Let* $k \geq 8$ *and* $s_k = 1 - \frac{3}{2(k-2)}$. *For any* $u_0 \in H^s(\mathbb{R}^2)$, $s > s_k$, *there exist* $T = T(\|u_0\|_{H^s}) > 0$ *and a unique solution of the IVP (22), defined in the interval* $[0, T]$, *such that*

$$
\begin{aligned}
& u \in C([0,T]; H^s(\mathbb{R}^2)), \\
& \|D_x^s u_x\|_{L_x^\infty L_{yT}^2} + \|D_y^s u_x\|_{L_x^\infty L_{yT}^2} < \infty, \\
& \|u\|_{L_T^{\widetilde{p}_k} L_{xy}^\infty} + \|u_x\|_{L_T^{12/5} L_{xy}^\infty} < \infty,
\end{aligned} \tag{25}
$$

and

$$\|u\|_{L_x^4 L_{yT}^\infty} < \infty, \tag{26}$$

where $\widetilde{p}_k = \frac{2(k-2)}{1-2\gamma}$ *and* $\gamma > 0$ *is sufficiently small. Moreover, for any* $T' \in (0, T)$ *there exists a neighborhood* U *of* u_0 *in* $H^s(\mathbb{R}^2)$ *such that the map* $\widetilde{u}_0 \mapsto \widetilde{u}(t)$ *from* U *into the class defined by (25)–(26) is smooth.*

Theorem 5 *(Farah-L-Pastor) Let $u_0 \in H^1(\mathbb{R})$. Let $k \geq 2$ and $s_k = (k-2)/k$. Suppose that*

$$E(u_0)^{s_k} M(u_0)^{1-s_k} < E(Q)^{s_k} M(Q)^{1-s_k}, \quad E(u_0) \geq 0. \tag{27}$$

If

$$\|\nabla u_0\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} < \|\nabla Q\|_{L^2}^{s_k} \|Q\|_{L^2}^{1-s_k}, \tag{28}$$

then for any t as long as the solution exists,

$$\|\nabla u(t)\|_{L^2}^{s_k} \|u_0\|_{L^2}^{1-s_k} = \|\nabla u(t)\|_{L^2}^{s_k} \|u(t)\|_{L^2}^{1-s_k} < \|\nabla Q\|_{L^2}^{s_k} \|Q\|_{L^2}^{1-s_k}, \tag{29}$$

where Q is the unique positive radial solution of (33). This in turn implies that H^1 solutions exist globally in time.

The main ingredient of the proof is the classical result obtained by M. Weinstein, regarding the best constant of the Gagliardo-Nirenberg inequality.

Theorem 6 *Let $k > 0$, then the Gagliardo-Nirenberg inequality*

$$\|u\|_{L^{k+2}}^{k+2} \leq K_{\text{opt}}^{k+2} \|\nabla u\|_{L^2}^{k} \|u\|_{L^2}^{2}, \tag{30}$$

holds, and the sharp constant $K_{\text{opt}} > 0$ is explicitly given by

$$K_{\text{opt}}^{k+2} = \frac{k+2}{2\|\psi\|_{L^2}^{k}}, \tag{31}$$

where ψ is the unique non-negative, radially-symmetric, decreasing solution of the equation

$$\frac{k}{2}\Delta\psi - \psi + \psi^{k+1} = 0. \tag{32}$$

Remark 1 *If ψ is the solution of (32), then by uniqueness*

$$Q(x,y) = \psi\left(\sqrt{\frac{k}{2}}(x,y)\right),$$

is the solution of

$$\Delta Q - Q + Q^{k+1} = 0. \tag{33}$$

Moreover,

$$\|Q\|_{L^2}^2 = \frac{2}{k}\|\psi\|_{L^2}^2.$$

In the critical case $k = 2$ we can go below $H^1(\mathbb{R}^2)$

Theorem 7 *(L-Pastor) Let* $k = 2$. *Let* $u_0 \in H^s(\mathbb{R}^2)$, $s > 53/63$, *and assume that* $\|u_0\|_{L^2} < \sqrt{3}\|\varphi\|_{L^2}$, *where* φ *is the ground state solution of equation* (33), *then the solution of* (22) *is globally well-posed.*

We use the low-high frequency method introduced by Bourgain.

Scattering

Theorem 8 *Let* $u_0 \in H^1(\mathbb{R}^2) \cap L^{p'}(\mathbb{R}^2)$ *such that*

$$\|u_0\|_{L^{p'}} + \|u_0\|_{H^1} < \delta.$$

where $p = 2(k+1)$, $p' = \frac{2(k+1)}{2k+1}$. *Let* $k \geq 3$ *and* $u(t)$ *be the global solution of* (22). *Then, there exist* $f_\pm \in H^1(\mathbb{R}^2)$ *such that*

$$\|u(t) - U(t)f_\pm\|_{H^1} \longrightarrow 0, \tag{34}$$

as $t \to \pm\infty$.

Remark 2 *Actually if* $k > \frac{3+\sqrt{33}}{4} \simeq 2.186$ *the result in Theorem* 8 *holds.*

Final Remarks

– We observe that Bourgain's approach to deal with the KdV equation, does not seem to work in our case. Indeed, it is well-known that to obtain "good bounds" by using the Fourier restriction method we need to know very well the behavior of the *resonant function*, or equivalently, the geometry of the *resonant set*, which is the zero set of the resonant function. In general, if the geometry of the resonant set is too "complicated" then it is not clear how to perform dyadic decompositions to get the needed estimates. This is the situation in our case where the resonant function is given by

$$h(\xi,\xi_1,\eta,\eta_1) = (\xi-\xi_1)(3\xi\xi_1+\eta\eta_1)+(\eta-\eta_1)(\xi\eta_1+\xi_1\eta).$$